UDC 531.36

ON THE ASYMPTOTIC STABILITY OF RESONANCE SYSTEMS

PMM Vol. 42, No. 6, 1978 pp 1033-1038

A. L. KUNITSYN
(Moscow)

(Received March 9, 1978)

Sufficient conditions of asymptotic stability as well as of instability derived directly from coefficients of normal form are presented.

The problem of stability of the trivial solution of the autonomous system of ordinary differential equations

$$dx / dt \equiv x^{\cdot} = X(x), \quad X(0) = 0, \quad x \in R^{2N} \tag{0.1}$$

is considered. Here $X(x)$ is a holomorphic vector function and matrix $(\partial X / \partial x)_{x=0}$ has only pure imaginary eigenvalues $\pm\lambda_s$ ($\lambda_s^2 < 0$; $s = 1, 2, \ldots, N$) that satisfy the condition of the fourth order inner resonance

$$\lambda_1 p_1 + \ldots + \lambda_N p_N = 0, \quad p_1 + \ldots + p_N = 4$$

where $p_1, \ldots, p_N$ are relatively prime nonnegative integers.

This problem was analyzed in detail only for Hamiltonian systems [1,2] for which the necessary and sufficient conditions of stability were obtained. For systems of the general form the problem is considerably more complicated, and may not be solvable algebraically even in the simplest cases (*). However it is important in applications to have at least sufficient but constructive criteria of stability and instability. A criterion of instability was proposed in [3].

1. It is shown in [4] that using Liapunov's normalizing transformation and polar coordinates r_j, θ_j ($j = 1, 2, \ldots, N$) it is possible to reduce the input system (0.1) to the form

$$\frac{1}{2} r_s^{\cdot} = \prod_{j=1}^{n} r_j^{p_j/2} Q_s(\theta) + r_s \sum_{\nu=1}^{N} a_{s\nu} r_\nu + \ldots \tag{1.1}$$

$$\theta^{\cdot} = \sum_{s=1}^{n} p_s \Bigl(\prod_{j=1}^{n} r_j^{p_j/2-\delta_{sj}} \frac{dQ_s}{d\theta} + \sum_{\nu=1}^{N} b_{s\nu} r_\nu \Bigr) + \ldots$$

$$s = 1, 2, \ldots, n; \ 2 \leqslant n \leqslant 4$$

$$r_\alpha^{\cdot} = 2 r_\alpha \sum_{\nu=1}^{N} a_{\alpha\nu} r_\nu + \ldots, \quad r_\alpha \theta_\alpha^{\cdot} = -i\lambda_\alpha r_\alpha + r_\alpha \sum_{\nu=1}^{N} b_{\alpha\nu} r_\nu + \ldots;$$

*) Shnol', E. E. and Khazin, L. G., The nonexistence of an algebraic criterion of asymptotic stability at resonance 1 : 3. Moscow, Preprint Inst. Matem. Akad. Nauk SSSR, No. 45, 1977.

$$\alpha = n+1, \ldots, N$$

$$\theta = p_1\theta_1 + \ldots + p_n\theta_n, \quad Q_s(\theta) = a_s \cos\theta + b_s \sin\theta, \quad i = \sqrt{-1}$$

where it is assumed that $p_{n+1} = \ldots = p_N = 0$, i.e. that only the part of eigenvalues $\lambda_1, \ldots, \lambda_n$ out of the total number N resonates, δ_{sj} is the Kronecker delta, and the omitted terms are of a higher order of smallness relative to $r_1, \ldots, r_N$.

When $a_s = b_s = 0$ we have the case of nonresonance, which was basically considered in [5, 6]. If, however, it is assumed that all $a_{s\nu} = b_{s\nu} = 0$, the problem is solved as in the case of third order resonance [4].

The aim of this investigation is to obtain sufficient conditions of asymptotic stability and instability in the general case, i.e. when both the terms of inner resonance $(a_s \neq 0,\ b_s \neq 0)$ and terms of the identical resonance $(a_{s\nu} \neq 0,\ b_{s\nu} \neq 0)$ are present.

Let us consider matrix

$$C = \begin{Vmatrix} a_1 & a_2 \ldots a_n \\ b_1 & b_2 \ldots b_n \end{Vmatrix}$$

and the corresponding to it all possible pairs of vectors $a = (a_\alpha, a_\beta, a_\gamma)$ and $b = (b_\alpha, b_\beta, b_\gamma)$, $\alpha, \beta, \gamma = 1, \ldots, n$, $\alpha \neq \beta \neq \gamma$, and $n > 2$. Let $D_{\beta\gamma}, D_{\gamma\alpha}, D_{\alpha\beta}$ be covariant components of the vector product $a \times b$ that are nonzero. The following lemma is then valid (an essentially similar statement appeared (without proof) in [5]).

L e m m a. For the system of linear equations

$$\sum_{s=1}^{n} a_s\gamma_s = 0, \quad \sum_{s=1}^{n} b_s\gamma_s = 0 \tag{1.2}$$

to have strictly positive (negative) solution for γ_s, it is necessary and sufficient that there exists at least one pair of vectors a and b for which there is no change of sign in the array of numbers $D_{\beta\gamma}$, $D_{\gamma\alpha}$, and $D_{\alpha\beta}$.

We shall prove the sufficiency by direct derivation of the indicated solution. It is evidently possible, without affecting the generality, to assume that, for instance, the pair of vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ satisfy the conditions of the lemma. Then from (1.2) we have

$$\gamma_1 = D_{12}^{-1}\Big(D_{23}\gamma_3 - \sum_{j=4}^{n} D_{j2}\gamma_j\Big), \quad \gamma_2 = D_{12}^{-1}\Big(D_{31}\gamma_3 - \sum_{j=4}^{n} D_{1j}\gamma_j\Big) \tag{1.3}$$

$$D_{\mu\nu} = \begin{vmatrix} a_\mu & a_\nu \\ b_\mu & b_\nu \end{vmatrix}$$

Owing to the arbitrariness of γ_3 and γ_j, they can be assumed positive. Subjecting these to inequalities

$$\Big|\sum_{j=4}^{n} D_{j2}\gamma_j\Big| < |D_{23}\gamma_3|, \quad \Big|\sum_{j=4}^{n} D_{1j}\Big| < |D_{13}\gamma_3| \tag{1.4}$$

we find that when the lemma is satisfied $\gamma_1 > 0$ and $\gamma_2 > 0$ which proves the sufficiency.

To prove the necessity we shall first show that when the two indicated vectors a and b are absent, the numbering of matrix C elements can be such that all determinants D_{1j} $(j = 2, \ldots, n)$ are of the same sign. For this we transpose the columns of matrix C so as to have

$$D_{1i} > 0,\ i = 2, \ldots, l;\ D_{1j} < 0,\ j = l + 1, \ldots, n$$

Owing to the nonfulfilment of conditions of the lemma for any pair of vectors a and b, it is possible to verify the validity of inequalities $D_{ij} < 0$ for $i = 1, \ldots, l;\ j = l + 1, \ldots, n$. Transposition of the first and l-th columns of matrix C yields on the one hand $D_{l1} < 0$ and, on the other maintains the negativeness of all determinants D_{lj} $(j = l + 1, \ldots, n)$. Carrying out a similar transposition the necessary number of times, we finally obtain $D_{1j} < 0$ $(j = 2, \ldots, n)$.

But, taking into account the property of determinants D_{1j} proved above, the second of Eqs. (1.3) shows that whatever the value of $\gamma_\alpha > 0$ $(\alpha = 3, \ldots, n)$ we always have $\gamma_2 < 0$, and consequently, system (1.2) has no strictly positive (negative) solution..

2. Let us consider the Liapunov function of the form

$$2V = \sum_{s=1}^{n} \gamma_s r_s + \sum_{\alpha=n+1}^{N} r_\alpha \tag{2.1}$$

where the constants γ_s satisfy Eqs. (1.2). If the conditions of the lemma proved above are satisfied all quantities γ_s can be assumed positive, and $n - 2$ of them must satisfy the two inequalities (1.4).

R e m a r k. For $n = 3$ no constraints are imposed on γ_3. For $n = 2$ the conditions of the lemma have no meaning, but, as can be readily seen, the necessary and sufficient conditions of existence of strictly positive (negative) solution of Eqs. (1.2) are of the form

$$a_1 / a_2 = -b_1 / b_2 < 0 \tag{2.2}$$

hence $\gamma_1 = - a_1 / a_2 \gamma_2$ and, consequently, one of the constants can be an arbitrary positive quantity.

Differentiating (2.1) on the strength of (1.1) and taking into account (1.2), we obtain

$$V^\cdot = \sum_{s=1}^{n} \gamma_s r_s (a_{s1} r_1 + \ldots + a_{sN} r_N) + \sum_{\alpha=n+1}^{N} r_\alpha (a_{\alpha 1} r_1 + \ldots + a_{\alpha N} r_N) + \ldots \tag{2.3}$$

where the order of smallness of omitted terms is not lower than the third with respect to variables $r_1, \ldots, r_N$ which solve the problem of stability.

Matrix M of quadratic form from which begins the expansion of (2.3) may be written in the form

$$M = \left\| \begin{array}{c:c} A_1 + A_1' & B' \\ \hdashline B & A_2 + A_2' \end{array} \right\| \tag{2.4}$$

$$A_1 = \| \gamma_i a_{ij} \|_{i,\,j=1}^{n}, \quad A_2 = \| a_{ij} \|_{i,\,j=n+1}^{N}$$

$$B = \| \gamma_i a_{ij} \|_{i=1,\,j=n+1}^{i=n,\,j=N} + \| a_{ij}' \|_{i=n+1,\,j=1}^{i=N,\,j=n}$$

where A_1 and A_2 are $n \times n$ and $(N-n) \times (N-n)$ square matrices, respectively, B is a rectangular matrix, and a prime denotes a transposed matrix.

On the basis of the above it possible to formulate the following theorem whose validity follows from Liapunov's theorem on asymptotic stability [7].

Theorem 1. Let system (1.1) be such that matrix C satisfies the condition of lemma. Then, if it is possible to select the $n-2$ $(n \neq 2)$ positive constants from among $\gamma_1, \ldots, \gamma_n$ satisfying Eqs. (1.2) so that the quadratic form in (2.3) is negative definite, then the trivial solution of system (1.1) is asymptotically stable.

Remark. For $n = 4$ the indicated constants must also satisfy inequalities (1.4), and for $n = 2$ condition (2.2) must be satisfied.

Using functions V of the form (2.1) it is possible to obtain also the sufficient conditions of instability. The latter can occur independently of whether the conditions of the lemma are satisfied or not. In particular the following theorem is valid.

Theorem 2. The trivial solution of system (1.1) is unstable when:

a) the condition of lemma is satisfied for matrix C and the constants $\gamma_1, \ldots, \gamma_n$ which satisfy Eqs. (1.2) can be selected so that the quadratic form in (2.3) is of constant sign ;

b) the condition of lemma is satisfied for matrix C and the above quadratic form can be made to be of fixed sign by [suitable] selection of $n-2$ positive constants from the totality of $\gamma_1, \ldots, \gamma_n$ that satisfy Eqs. (1.2)

c) the condition of lemma is satisfied for matrix C and the above quadratic form can be made positive definite by suitable selection of $n-2$ positive constants from the totality $\gamma_1, \ldots, \gamma_n$ that satisfy Eqs. (1.2).

In each of the above cases function V of the form (1.2) satisfies the conditions of Liapunov's theorem on instability [7]. Thus, as implied by the lemma, constants $\gamma_1, \ldots, \gamma_n$ cannot in case a) be simultaneously positive and, consequently, V is of alternating sign. The same applies to V in case b), since when conditions of the lemma and inequalities (1.4) are satisfied, all $\gamma_1, \ldots, \gamma_n$ can be made negative.

Finally, in case c) function V and its derivative are positive definite.

3. To elucidate the constructiveness of the obtained criterion we consider certain particular cases of system (1.1).

Case 1. We set in (1.1) $n = N = 2$. Equations (1.1) may then define, for instance, the perturbed motion of a non-Hamiltonian mechanical system with two degrees of freedom.

The conditions of strict positiveness (negativeness) of solution of system (1.2) are of the form (2.2), hence one of the constants γ_1 or γ_2 can be an arbitrary positive quantity. If one succeeds in making matrix M (2.4) positive definite by the selection of that constant, function V of the form (2.1) satisfies then Liapunov's theorem on asymptotic stability.

For $\gamma_1 > 0$ and $\gamma_2 > 0$ the necessary and sufficient conditions of negativeness of matrix M in the cone $r_1 \geqslant 0$ and $r_2 \geqslant 0$ are of the form

$$a_{11} < 0, \quad a_{12} < \frac{a_1}{a_2} a_{21} + 2 \sqrt{-\frac{a_1}{a_2} a_{11} a_{22}} \tag{3.1}$$

and V and $V^{\bullet}$ are then of constant but different signs. Hence when (2.2) and (3.1) are satisfied, the trivial solution is asymptotically stable.

If we set $\gamma_1 > 0$ $(\gamma_2 < 0)$ and reverse the inequality sign of the first inequality, then when (2.2) is satisfied V and $V^{\bullet}$ are negative definite and the trivial solution is unstable. It is also unstable for any sign of a_{11}, if for $-a_1/a_2 = b_1/b_2 < 0$ either the second of inequalities (3.1) or the inequality

$$a_{12} > \frac{a_1}{a_2} a_{21} - 2\left(-\frac{a_1}{a_2} a_{11} a_{22}\right)^{1/2}$$

is satisfied, since in both cases the derivative $V^{\bullet}$ is positive or negative definite, while function V itself is of alternating sign because $\gamma_1\gamma_2 < 0$.

An interesting peculiarity of the class of systems considered above (distinguished by condition (2.2)) is that in it an asymptotic stability is possible for as large as desired values of coefficients at the resonant terms a_s and b_s. This is implied by the second of inequalities (3.1) if we set in it

$$a_s = \sqrt{a_s^2 + b_s^2} \sin\psi_s, \quad b_s = \sqrt{a_s^2 + b_s^2} \cos\psi_s, \quad s = 1, 2$$

C a s e 2. We now set in (1.1) $n = N = 3$, i.e. we consider a system with three degrees of freedom. We set the matrix M(2.4)

$$M = \begin{Vmatrix} 2\gamma_1 a_{11} & \gamma_1 a_{12} + \gamma_2 a_{21} & \gamma_1 a_{13} + \gamma_3 a_{31} \\ \gamma_1 a_{12} + \gamma_2 a_{21} & 2\gamma_2 a_{22} & \gamma_2 a_{23} + \gamma_3 a_{32} \\ \gamma_1 a_{13} + \gamma_3 a_{31} & \gamma_2 a_{23} + \gamma_3 a_{32} & 2\gamma_3 a_{33} \end{Vmatrix}$$

where, on the basis of (1.3)

$$\gamma_1 = \frac{D_{23}}{D_{12}} \gamma_3, \quad \gamma_2 = \frac{D_{31}}{D_{12}} \gamma_3$$

where γ_3 is an arbitrary constant which can be assumed positive. If coefficients a_s and b_s satisfy the conditions of the lemma, we have $\gamma_1 > 0$ and $\gamma_2 > 0$.

On the strength of the Sylvester criterion we obtain the following conditions for matrix M to be negative definite:

$$\begin{gathered} a_{11} > 0, \quad F \equiv 4D_{23}D_{31}a_{11}a_{22} - \delta_1^2 > 0 \\ G \equiv Fa_{33} - \frac{D_{23}}{D_{12}} a_{11}\delta_3^2 + \delta_1\delta_2\delta_3 - \frac{D_{31}}{D_{12}} a_{22}\delta_2^2 < 0 \\ \delta_1 = D_{23}a_{12} + D_{31}a_{21}, \quad \delta_2 = D_{23}a_{13} + D_{12}a_{31} \end{gathered} \tag{3.2}$$

$$\delta_3 = D_{23}a_{13} + D_{12}a_{31}$$

We conclude on the basis of Theorem 1 that inequalities (3.1) together with conditions of the lemma represent sufficient conditions of asymptotic stability. The obtained inequalities are obviously compatible, since coefficient a_{11} does not appear in any of the last inequalities in (3.2) and the third inequality contains a term with coefficients that are absent in the second inequality. It is, thus, possible to obtain for a sixth order system the conditions of asymptotic stability without having to select a suitable value for the arbitrary constant γ_3.

Reversal of the signs of the first and last of inequalities (3.2) results, in virtue of Theorem 2, in an unstable trivial solution of system (1.1) (case c). It is evident that the trivial solution is also unstable (independently of whether conditions of the lemma are satisfied or not), if only the inequalities

$$F > 0, \ a_{11}G < 0$$

are satisfied, since then cases a) and b) of Theorem 2 are valid.

Note that conditions (3.2) may be, generally speaking, widened, since it is sufficient to specify that $V^{\cdot}$ must be of definite sign only in the positive cone $r_s \geqslant 0$ $(s = 1, 2, \ldots, n)$. However the analytic representation in the case of $n \geqslant 3$ is a very difficult problem which reduces to the question of compatibility of the system of inequalities [8].

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Translated by J.J. D.